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A MODEL OF FRACTURING MATERIAL WITH DILATANCY

N. G. Bourago and A. N. Kovshov

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This work deals with constitutive relations describing the thermomechanical behavior of geomaterials. A general thermodynamical method for the construction of constitutive relations for continuous media with finitely many state parameters is described in [1] and is a generalization of the investigations started in [2–5]. In the present paper, the thermodynamical method is systematically applied to modeling deformation of geomaterials. In contrast to previous publications, our construction of the model requires only that the free energy and the dissipation velocity be specified as functions of state parameters. Then, the dilatancy and other properties of geomaterials become mere consequences.

Well known experimental studies show that geomaterials possess some distinctive properties, such as plastic compressibility and non-monotonicity of the "shear versus tangential stress" loading diagram which, together with a hardening segment contains a "dropping" segment corresponding to material softening. Fracture of geomaterials occurs gradually, with the accumulation of various types of microdefects; moreover, wave propagation velocities in geomaterials depend on the damage measure and become lower with its increase.

As shown by numerous experiments, a characteristic feature of geomaterials is their dilatancy, i.e., the existence of a dependence of volume strain on shear strains.

The dilatancy of geomaterials can be described in many ways. As shown in [6], if the deformation of a geomaterial is described by the equations of plastic flow with the associated law and the yield condition of the von Mises type with the yield function depending on the first invariant of the stress tensor, then dilatancy takes place. Moreover, the mechanical energy dissipation rate and the volume expansion rate in the case of plane deformation are proportional to the maximal shear rate, with the coefficient of proportionality equal to the internal friction coefficient.

This relation implies that there is a dependence between the components of the plastic strain rate tensor, and this dependence is called the dilatancy condition or the dilatancy dependence. In [7–9], the dilatancy condition is taken as an assumption and has the form of a kinematic constraint on the plastic strain rate components; this constraint involves the so-called dilatancy rate. As a result, the sign of the dilatancy rate determines the possibility of loosening or compaction of the dilating geomaterial under shear. Moreover, the constraint relating the density of the medium and the pressure p is differential and this differential constraint is not always integrable. The function $p(\rho)$ may be different for different deformation paths of the material elements.

The possibility of constructing models of plastic deformation of materials (in particular, loose materials) on the basis of the properties of the dissipation function was considered in [10, 11]. It was shown that for a prescribed dilatancy dependence, the dissipation function can be defined in such a way that the constitutive relations for a medium with dilatancy can be introduced by means of the associated law. The dilatancy dependence on the form of the stress state is studied in [12], where the associated plastic flow law is used with a suitably chosen structure of the yield function and the dilatancy condition is not prescribed in advance.

Thus, in order to describe the behavior of geomaterials, various models have been proposed, with the above properties more or less taken into account. In the so-called models of "gradual fracture", the constitutive relations rely on the associated elastic-plastic flow law and the yield surface is specified so as to take into account the phenomena of hardening, softening, and "residual" strength. In continuous fracture models, which are currently an object of intensive studies, fracture of geomaterials is described in terms of a new structural parameter referred to as the damage parameter. This parameter pertains to the accumulation of microdefects in a deformed medium and is responsible for the decrease of wave propagation velocities in geomaterials. In the models of geomaterials with damage, which seem physically most adequate, damage accumulation rate is introduced and its backward influence on the stress-strain state is taken into account in terms of the dependence of elastic and plastic properties on the damage. The constitutive relations in damage models are often constructed on the basis of micromechanical concepts regarding the processes of deformation and fracture in geomaterials.

The constitutive relations should agree with the general principles of thermomechanics, i.e., be compatible with the laws of thermodynamics, the dimensional theory, and the principles of invariance and objectivity. These requirements can be easily met when constructing the constitutive relations by the thermodynamical method [1] used below.

Keeping in mind that deformation of geomaterials is accompanied by temperature variations, heat transfer, elastic-plastic flow with dilatancy, and damage accumulation, we consider the following independent state parameters: temperature T; material strain tensor $\mathring{\epsilon}$, material plastic strain tensor $\mathring{\epsilon}_{\rm p}$; damage ω; their rates, dT/dt, $\mathring{\mathbf{e}} = d\mathring{\epsilon}/dt$, $d\omega/dt$, and temperature gradient ∇T.

We assume that the free energy φ and the dissipation rate function D (dissipation function) are functions of these basic parameters. It is necessary to specify the structure of these functions on the basis of the notion concerning physical properties of geomaterials.

We consider the plastic strain tensor $\mathring{\epsilon}_{\rm p}$ as the structural parameter responsible for changes in the material structure due to plastic deformation. This approach is somewhat unusual but has the advantage of allowing us to avoid a separate definition of the elastic strain tensor, instead of which we use the difference between the strain tensor $\mathring{\epsilon}$ and the plastic strain tensor $\mathring{\epsilon}_{\rm p}$. Moreover, in the strain space, the plastic strain tensor $\mathring{\epsilon}_{\rm p}$ specifies a point at which the energy attains its minimum with respect to strains.

The free energy φ is a function of temperature T, strain tensor $\mathring{\epsilon}$, plastic strain tensor $\mathring{\epsilon}_{\rm p}$, and damage measure ω. The independence of the free energy of the remaining parameters is established in [1]. The reversible volume strain is not very large for geomaterials under moderate loads. Elastic shear strains, too, cannot be very large. These assumptions allow us to take the following simple expression for the free energy:

$$\varphi = \frac{K(T,\omega)}{2\rho}\left((\epsilon - \epsilon_{\rm p}) \cdot\cdot\, \mathbf{I}\right)^2 + \frac{\mu(T,\omega)}{\rho}\left((\epsilon' - \epsilon'_{\rm p}) \cdot\cdot\, (\epsilon' - \epsilon'_{\rm p})\right), \tag{1}$$

where ρ is the current density; $K(T,\omega)$ is the bulk modulus depending on the temperature and the damage ω; $\mu(T,\omega)$ is the shear modulus; ϵ' is the deviator of the strain tensor; $\epsilon'_{\rm p}$ is the deviator of the plastic strain tensor; $\mathbf{I}$ is the identity tensor; the symbol $\cdot\cdot$ denotes the scalar product of tensors. The spatial strain tensor ϵ and the spatial plastic strain tensor $\epsilon_{\rm p}$ in this expression for the free energy are related to the corresponding material tensors by

$$\epsilon = \mathbf{F}^{-\rm T} \cdot \mathring{\epsilon} \cdot \mathbf{F}^{-1}, \tag{2}$$

$$\epsilon_{\rm p} = \mathbf{F}^{-\rm T} \cdot \mathring{\epsilon}_{\rm p} \cdot \mathbf{F}^{-1}, \tag{3}$$

where $\mathbf{F}$ is the deformation gradient.

The structure of the dissipation rate function D should allow us to describe the plastic volume compressibility, viscoplastic properties of geomaterials, their heat conductivity properties, and also damage accumulation and dilatancy. We take the function D in the form

$$D = H(A_1)k_{\rm v}f_{\rm v}(\mathbf{e}_{\rm p} \cdot\cdot\, \mathbf{I})^2 + H(A_2)k_{\rm p}f_{\rm p}(\mathbf{e}'_{\rm p} \cdot\cdot\, \mathbf{e}'_{\rm p}) + H(\Phi)k_\omega\left(\frac{d\omega}{dt}\right)^2 + \frac{k_{\rm q}}{T}\nabla T \cdot \nabla T, \tag{4}$$

where H is the Heavyside unit step function equal to unity for the nonnegative values of its argument and equal to zero elsewhere. The arguments A_1 and A_2 determine the plasticity conditions with respect to the volume and the shear,

$$A_1 = p^2 - k_{\rm v}^2 f_{\rm v}^2(\mathbf{e}_{\rm p} \cdot\cdot\, \mathbf{I})^2, \quad A_2 = \boldsymbol{\sigma}' \cdot\cdot\, \boldsymbol{\sigma}' - k_{\rm p}^2 f_{\rm p}^2(\mathbf{e}'_{\rm p} \cdot\cdot\, \mathbf{e}'_{\rm p});$$

where $p = -(\boldsymbol{\sigma} \cdot\cdot\, \mathbf{I})/3$ is pressure and $\boldsymbol{\sigma}'$ is the deviator of the Cauchy stress tensor. It is assumed that the damage measure ω is a scalar quantity which grows if the following fracture criterion holds:

$$\Phi(T, \epsilon, \epsilon_{\rm p}, \omega) \geq 0. \tag{5}$$

The variables $f_{\rm v}$ and $f_{\rm p}$ are positive functions of the invariants of the tensor and/or the deviator of plastic strain rates, respectively. The coefficients $k_{\rm v}$, $k_{\rm p}$, k_ω, $k_{\rm q}$ are positive functions of the temperature, the strains, and the structural parameters $\epsilon_{\rm p}$, ω.

The first term in the expression for dissipation rate (4) describes the irreversible viscoplastic volume deformation; the second term describes the irreversible viscoplastic shape variation; the third term corresponds to damage accumulation ω, and the last term is connected with the heat transfer. Thus, the dissipation function is determined by the following variables: the tensor of plastic strains $\boldsymbol{\epsilon}_p$, which is regarded as a structural parameter and enters the dissipation function through the plastic strain rate tensor $\mathbf{e}_p$ and its deviator $\mathbf{e}'_p$; the structural parameter of damage, ω; and the temperature gradient ∇T.

Note also that f_v and f_p are not necessarily homogeneous functions of their arguments. Therefore, the dissipation function may also be nonhomogeneous. Here, in contrast to [13], no additional extremum principles are needed for the derivation of constitutive relations.

The desired constitutive relations are found simply as a possible solution of the dissipation inequality

$$-\rho\left(\eta+\frac{\partial\varphi}{\partial T}\right)\frac{dT}{dt}+\left(\mathring{\boldsymbol{\sigma}}-\rho\frac{\partial\varphi}{\partial\mathring{\boldsymbol{\epsilon}}}\right)\cdot\cdot\,\mathring{\mathbf{e}}-\rho\left(\frac{\partial\varphi}{\partial\mathring{\boldsymbol{\epsilon}}_p}\right)\cdot\cdot\,\mathring{\mathbf{e}}_p-\rho\frac{\partial\varphi}{\partial\omega}\frac{d\omega}{dt}+\mathbf{q}\cdot\frac{\nabla T}{T}\geq 0$$

and have the form

$$\eta+\frac{\partial\varphi}{\partial T}=\eta_D(\pi),\quad \mathring{\boldsymbol{\sigma}}-\rho\frac{\partial\varphi}{\partial\mathring{\boldsymbol{\epsilon}}}=\mathring{\boldsymbol{\sigma}}_D(\pi),\quad -\frac{\partial\varphi}{\partial\mathring{\chi}}=\mathring{\mathbf{X}}_D(\pi),\quad \mathbf{q}=\mathbf{q}_D(\pi),$$

where the right-hand sides determine the dissipation rate,

$$D=-\rho\eta_D\frac{dT}{dt}+\mathring{\boldsymbol{\sigma}}_D\cdot\cdot\,\mathring{\mathbf{e}}-\rho\mathring{\mathbf{X}}_D\cdot\cdot\,\frac{d\mathring{\chi}}{dt}+\mathbf{q}_D\cdot\frac{\nabla T}{T}\geq 0,$$

and are found (calculated) according to the expression for the dissipation rate (4). We have also used the following notation:

$$\pi=\left(T,\mathring{\boldsymbol{\epsilon}},\mathring{\chi},\frac{dT}{dt},\mathring{\mathbf{e}},\frac{d\mathring{\chi}}{dt},\nabla T\right),\quad \mathring{\chi}=(\mathring{\boldsymbol{\epsilon}}_p,\omega),\quad \mathring{\boldsymbol{\sigma}}=\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}}.$$

Taking into account the above expression for the free energy and the fact that the dissipation function does not depend on the strain rate tensor ($\mathring{\boldsymbol{\sigma}}_D=0$), we obtain constitutive relations for stresses in the form

$$\mathring{\boldsymbol{\sigma}}=\rho\frac{\partial\varphi}{\partial\mathring{\boldsymbol{\epsilon}}},\tag{6}$$

or

$$-p\mathbf{I}+\boldsymbol{\sigma}'=K\big((\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)\cdot\cdot\,\mathbf{I}\big)\mathbf{I}+2\mu(\boldsymbol{\epsilon}'-\boldsymbol{\epsilon}'_p),\tag{7}$$

where $p=-(\boldsymbol{\sigma}\cdot\cdot\,\mathbf{I})/3$ is pressure and $\boldsymbol{\sigma}'$ is the deviator of the stress tensor. It follows that

$$p=-K(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)\cdot\cdot\,\mathbf{I},\quad \boldsymbol{\sigma}'=2\mu(\boldsymbol{\epsilon}'-\boldsymbol{\epsilon}'_p).$$

The constitutive relations for the deviatoric and the spherical parts of the plastic strain rate tensor $\mathbf{e}_p$ are obtained in the form

$$\mathbf{e}'_p=\frac{H(A_2)}{k_p f_p}\boldsymbol{\sigma}',\tag{8}$$

$$(\mathbf{e}_p\cdot\cdot\,\mathbf{I})=-\frac{H(A_1)}{k_v f_v}p.\tag{9}$$

In a similar way we obtain the constitutive equation for damage ω,

$$k_\omega\frac{d\omega}{dt}=-H(\Phi)\left[\frac{1}{2}\big((\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)\cdot\cdot\,\mathbf{I}\big)^2\frac{\partial K}{\partial\omega}+(\boldsymbol{\epsilon}'-\boldsymbol{\epsilon}_p)^2\cdot\cdot\,\mathbf{I}\frac{\partial\mu}{\partial\omega}\right].\tag{10}$$

The constitutive relations for the entropy, the internal energy, and the heat flows have the form

$$\eta=-\frac{\partial\varphi}{\partial T},\quad \eta_D=0,\quad U=\varphi+\eta T,\quad \mathbf{q}=\mathbf{q}_D=k_q\nabla T.\tag{11}$$

Let us examine the above constitutive relations. For $\partial K/\partial\omega \le 0$ and $\partial\mu/\partial\omega \le 0$, it follows from (10) that

$$k_\omega \frac{\partial \omega}{\partial t} \ge 0, \tag{12}$$

and thus damage is described by a non-decreasing function of time, which agrees with physical considerations. In order that damage may grow, the condition of fracture $\Phi \ge 0$ should be satisfied. The decrease of wave propagation velocities with damage growth is ensured by the decrease of the elastic moduli K and μ with the growth of damage.

Consider some dilatancy effects. Relation (9), in general, serves as a dilatancy relation. In the special case, if we assume that

$$f_u = (\mathbf{e}'_\mathrm{p} \cdot\cdot\, \mathbf{e}'_\mathrm{p})^{-1/2}, \tag{13}$$

we obtain the following dilatancy relation for the expansion rate from (9):

$$\mathbf{e}_\mathrm{p} \cdot\cdot\, \mathbf{I} = -H(A_1)\frac{p}{k_\mathrm{v}}(\mathbf{e}_\mathrm{p} \cdot\cdot\, \mathbf{e}_\mathrm{p})^{1/2}. \tag{14}$$

Relation (14) is a generalization of a result of [6]. From (14) in the case of plane deformation, just as in [6], it follows that the expansion rate is proportional the maximal shear rate. But, in contrast to [6], the coefficient of proportionality depends on the pressure. The dependence of the volume expansion rate on the shear rate means that the proposed model describes the dilatancy phenomenon observed in geomaterials subject to deformation. The phenomenon of plastic volume compressibility is also described by (9). It can be seen that positive pressure results in compression. The inequality $A_1 > 0$ determines the plasticity condition with respect to volume.

The non-monotonicity of the shear loading diagram can be described by (8), provided that the forming loading condition ("the shear plasticity condition") $A_2 > 0$ is properly specified.

The problem of specification of loading functions $A_1(T, \epsilon, \epsilon_\mathrm{p}, \omega)$ and $A_2(T, \epsilon, \epsilon_\mathrm{p}, \omega)$ and the fracture function $\Phi(T, \epsilon, \epsilon_\mathrm{p}, \omega)$ requires a separate discussion (a detailed review of these topics can be found in [5]). This question reflects one of the main problems in the mechanics of plastic media and is beyond the scope of the present paper.

In this paper, we have obtained constitutive relations for a dilatable elastoviscoplastic medium subject to fracture. These relations have been suggested by the laws of thermodynamics. Of course, these relations cannot be claimed most general, but they represent one of the possible models. The range of approaches to the construction of constitutive relations is very wide. For instance, when writing the dissipation function, we could have explicitly used other invariants of the plastic strain rate tensor. Instead of (or together with) the volume plastic strain, we could have used porosity or residual density (the density after load-relief) as a state parameter, which in some cases is more convenient (see, for instance, [1]), especially for the analysis of finite deformations. Any modification, specification, or calibration of constitutive relations should be performed in close connection with physical experiments and should take into account specific properties of geomaterials. Above, we have merely demonstrated a simple method for the construction of such relations.

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