Int. J. Comp. Civil & Structural Eng. Vol.1 (2004) issue 4 pp. 0000 (© Begell House 2004)

SURFACE WAVE ON A HALF-SPACE WITH CUBIC SYMMETRY

By

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KEYWORDS: surface wave, Rayleigh wave, anisotropy, cubic crystal

ABSTRACT: In a previous publication [1] existence of "forbidden" planes for some transversely isotropic half spaces upon which the genuine Rayleigh waves cannot propagate was established. Now, it is shown that for specific cubic crystals and the directions of elastic symmetry there arise exponentially attenuating with depth surface waves of the non-Rayleigh type.

1.INTRODUCTION

In our previous paper [1] it was shown that some transversely isotropic media exhibit property of non-existence of the genuine Rayleigh waves. The latter can be defined by the following expression

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{3} C_k \mathbf{m}_k e^{ir(\gamma_k \boldsymbol{\nu}\cdot\mathbf{x} + \mathbf{n}\cdot\mathbf{x} - ct)} \quad (1.1)$$

where C_k are complex coefficients determined up to a multiplier by the traction-free boundary conditions; $\mathbf{m}_k$ are complex eigenvectors of the Christoffel equation, which will be introduced further; these eigenvectors correspond to complex roots γ_k of the characteristic polynomial; r is the (real) wave number; $\boldsymbol{\nu}$ is an outward normal to the boundary $\Pi_{\boldsymbol{\nu}}$ of the half-space along which the surface wave propagates; $\mathbf{n} \in \Pi_{\boldsymbol{\nu}}$ is the unit vector determining direction of propagation of the surface wave, and c is the phase speed. The terms

$$\mathbf{u}_k(\mathbf{x}) \equiv \mathbf{m}_k e^{ir(\gamma_k \boldsymbol{\nu}\cdot\mathbf{x} + \mathbf{n}\cdot\mathbf{x} - ct)} \quad (1.2)$$

are called *partial waves*.

As was shown in [1], the existence of the "forbidden" planes upon which the genuine Rayleigh wave cannot propagate is due to appearing the Jordan blocks in a specially constructed 6×6-matrix associated with the Christoffel equation.

The following analysis reveals that the situation regarded in [1] appears to be more complicated. The Jordan blocks in the regarded matrix lead to a qualitative change of the structure of the partial waves (1.2) and, while the genuine Rayleigh wave (1.1) at the situation considered in [1] does not exist, there remains an exponentially attenuating with depth surface wave of the non-Rayleigh type. It should also be noted that some surface waves of the non-Rayleigh type were reported in [3-5] by applying Stroh's sextic formalism.

2 BASIC NOTATIONS

Equations of motion for an anisotropic elastic medium can be written in the form

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \operatorname{div}_x \mathbf{C}\cdot\cdot\nabla_x \mathbf{u} - \rho\ddot{\mathbf{u}} = 0 \quad (2.1)$$

where $\mathbf{u}$ is the displacement field; ρ is the density of a medium; and $\mathbf{C}$ is the fourth-order elasticity tensor assumed to be *positive definite*:

$$\forall_{\mathbf{A}\in sym(R^3\otimes R^3),\ \mathbf{A}\neq 0} \mathbf{A} \quad (\mathbf{A}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{A}) > 0 \quad (2.2)$$

The sign "$\cdot$" in (2.1), (2.2) and henceforth means the scalar multiplication in the corresponding unitary or Euclidian vector space:

$$\mathbf{a}\cdot\mathbf{b} = \sum_k a_k \overline{b^k} \quad (2.3)$$

Substituting partial waves (1.2) in Eq. (2.1) produces the Christoffel equation:

$$\left[(\gamma_k \boldsymbol{\nu} + \mathbf{n})\cdot\mathbf{C}\cdot(\mathbf{n} + \gamma_k\boldsymbol{\nu}) - \rho c^2 \mathbf{I} \right]\cdot\mathbf{m}_k = 0$$

where $\mathbf{I}$ is the unit diagonal matrix. Equation (2.3) can be written in the equivalent form:

$$\det\left[(\gamma_k \boldsymbol{\nu} + \mathbf{n})\cdot \mathbf{C}\cdot(\mathbf{n}+\gamma_k\boldsymbol{\nu}) - \rho c^2 \mathbf{I}\right] = 0 \quad (2.4')$$

The left-hand side of Eq. (2.4′) represents a polynomial of degree 6 with respect to γ_k.

REMARK 2.1. It can be shown, see [1], that if the phase speed does not exceed the so called lower limiting speed ($c_3^{\lim}$):

$$c < c_3^{\lim}, \qquad (2.5)$$

then all the roots of Eq. (2.3) are complex with $\operatorname{Im}(\gamma_k) \neq 0$. The inequality (2.5) ensures that there exist three (not necessary aliquant) roots γ_k with $\operatorname{Im}(\gamma_k) < 0$, which ensure attenuation with depth in a "lower" half-space at $(\boldsymbol{\nu}\cdot\mathbf{x}) < 0$. Only attenuating with depth partial waves, as being physically reasonable, will be considered.

3 SIX-DIMENSIONAL FORMALISM

Following [1], a more general representation for the partial wave than (1.2), will be considered:

$$\mathbf{v}(x'') e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)} \qquad (3.1)$$

where $x'' = ir\,\boldsymbol{\nu}\cdot\mathbf{x}$ is the dimensionless complex coordinate, $\mathbf{v}(x'')$ is an unknown vector function, and the exponential multiplier in (3.1) corresponds to propagation of the plane wave front along the direction $\mathbf{n}$ with the phase speed c. Substituting representation (3.1) into Eq. (2.1) yields the following system of ordinary differential equations:

$$\begin{pmatrix} (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\partial_{x''}^2 + (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\partial_{x''} - \\ (\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2 \mathbf{I}\ ) \end{pmatrix} \mathbf{v}(x'') = 0 \qquad (3.2)$$

Direct analysis of system (3.2) is rather difficult, and reduction to the first-order system can considerably simplify it.

Introduction of a new vector-function $\mathbf{w} = \partial_{x''}\mathbf{v}$ allows us to reduce the second-order system (3.2) in C^3 to the first-order one in C^6:

$$\partial_{x''}\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix} \qquad (3.3)$$

In (3.3) the complex six-dimensional matrix $\mathbf{R}_6$ has the form

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M} & -\mathbf{N} \end{pmatrix} \qquad (3.4)$$

where three-dimensional matrices $\mathbf{M}$ and $\mathbf{N}$ have the form

$$\begin{aligned} \mathbf{M} &= (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})^{-1}\cdot(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I}\ ) \\ \mathbf{N} &= (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})^{-1}\cdot(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) \end{aligned} \qquad (3.5)$$

In (3.4) $\mathbf{I}$ stands for the unit (diagonal) matrix in the three-dimensional space.

A surjective homomorphism $\Im :\ C^6 \to C^3$, such that

$$\Im(\mathbf{v}, \mathbf{w}) = \mathbf{v} \qquad (3.6)$$

will be needed for the subsequent analysis.

The following Proposition takes place [1]:

PROPOSITION 3.1. Let $c \in (0;\ \ c_3^{\lim})$:

a) Spectrum of the matrix $\mathbf{R}_6$ coincides with the set of all roots of polynomial (2.4);

b) If γ is a complex eigenvalue and $\mathbf{m} = (\mathbf{m}',\ \mathbf{m}'')$, $\mathbf{m}',\ \mathbf{m}'' \in C^3$ is the corresponding six-dimensional eigenvector of the matrix $\mathbf{R}_6$, then $\overline{\gamma}$ is also an eigenvalue with the corresponding eigenvector $\overline{\mathbf{m}} = (\overline{\mathbf{m}'},\ \overline{\mathbf{m}''})$;

c) The matrix $\mathbf{R}_6$ admits the following Jordan normal forms

$$\mathbf{J}_6^{(\mathrm{I})} = \operatorname{diag}\left(\gamma_1, \overline{\gamma}_1, \gamma_2, \overline{\gamma}_2, \gamma_3, \overline{\gamma}_3\right),$$

$$\mathbf{J}_6^{(\mathrm{II})} = \operatorname{diag}\left(\begin{pmatrix}\gamma_1 & 1\\ 0 & \gamma_1\end{pmatrix}, \begin{pmatrix}\overline{\gamma}_1 & 1\\ 0 & \overline{\gamma}_1\end{pmatrix}, \gamma_3, \overline{\gamma}_3\right), \qquad (3.7)$$

$$\mathbf{J}_6^{(\mathrm{III})} = \operatorname{diag}\left(\begin{pmatrix}\gamma_1 & 1 & 0\\ 0 & \gamma_1 & 1\\ 0 & 0 & \gamma_1\end{pmatrix}, \begin{pmatrix}\overline{\gamma}_1 & 1 & 0\\ 0 & \overline{\gamma}_1 & 1\\ 0 & 0 & \overline{\gamma}_1\end{pmatrix}\right)$$

d) According to the Jordan normal forms the following three types of representations for surface waves occur:

(i) for the Jordan normal form $\mathbf{J}_6^{(\mathrm{I})}$, the corresponding representation is given by (1.1);

(ii) for the Jordan normal form $\mathbf{J}_6^{(\mathrm{II})}$, the representation is as follows:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = (C_1 + ir\,C_2\,\boldsymbol{\nu}\cdot\mathbf{x})\,\mathbf{m}_1'\,e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} \\ + C_2\,\mathbf{m}_2'\,e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} \\ + C_3\,\mathbf{m}_3'\,e^{ir(\gamma_3\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)}\end{aligned} \tag{3.8}$$

where $\mathbf{m}_1' = \Im(\mathbf{m}_1) \in C^3$, and $\mathbf{m}_1$ is the eigenvector of $\mathbf{R}_6$ corresponding to the eigenvalue γ_1; $\mathbf{m}_2' = \Im(\mathbf{m}_2) \in C^3$, and $\mathbf{m}_2 \in C^3$ is the generalized eigenvector associated with $\mathbf{m}_1$, and the eigenvector $\mathbf{m}_3 \in C^6$ corresponds to the eigenvalue γ_3;

(iii) for the Jordan normal form $\mathbf{J}_6^{(\mathrm{III})}$, the representation is as follows:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = (C_1 + ir\,C_2\boldsymbol{\nu}\cdot\mathbf{x} + \tfrac{1}{2}C_3\,(ir\,\boldsymbol{\nu}\cdot\mathbf{x})^2)\times \\ \mathbf{m}_1'\,e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} + \\ (C_2 + ir\,C_3\boldsymbol{\nu}\cdot\mathbf{x})\,\mathbf{m}_2'\,e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)} + \\ C_3\,\mathbf{m}_3'\,e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x}-ct)}\end{aligned} \tag{3.9}$$

$\mathbf{m}_1' = \Im(\mathbf{m}_1) \in C^3$, and $\mathbf{m}_1$ is the eigenvector corresponding to the eigenvalue γ_1; and $\mathbf{m}_2, \mathbf{m}_3 \in C^6$ are the generalized eigenvectors associated with $\mathbf{m}_1$.

COROLLARY. For any of the Jordan normal forms of the matrix $\mathbf{R}_6$ the three-dimensional components $\mathbf{m}_k', \mathbf{m}_k''$ of the (proper) eigenvector $\mathbf{m}_k$, satisfy the equations

$$\begin{aligned}&\mathbf{m}_k'' = \gamma_k\,\mathbf{m}_k' \\ &\left(\gamma_k^2\mathbf{I} + \gamma_k\mathbf{N} + \mathbf{M}\right)\cdot\mathbf{m}_k' = 0\end{aligned} \tag{3.10}$$

Proof. When the matrix $\mathbf{R}_6$ has no Jordan blocks, the solution of Eq. (3.3) in view of (3.4) leads to Eqs. (3.10). Thus, the component $\mathbf{m}_k'$ belongs to the kernel space of the matrix $\left(\gamma_k^2\mathbf{I} + \gamma_k\mathbf{N} + \mathbf{M}\right)$. The analogous situation takes place for any of the proper eigenvectors.

4 CONSTRUCTING THE GENERALIZED EIGENVECTOR FOR $\mathbf{J}_6^{(\mathrm{II})}$

In view of [2] the solution of Eq. (3.3) corresponding to a Jordan block of the second rank can be represented in the form

$$\begin{pmatrix} C_1\left(\mathbf{m}_1', \mathbf{m}_1''\right) + \\ C_2\left(x''\left(\mathbf{m}_1', \mathbf{m}_1''\right) + \left(\mathbf{m}_2', \mathbf{m}_2''\right)\right)\end{pmatrix} e^{\gamma_1 x''} \tag{4.1}$$

where as before $x'' = ir\,\boldsymbol{\nu}\cdot\mathbf{x}$.

PROPOSITION 4.1. a) The three-dimensional components $\mathbf{m}_1', \mathbf{m}_1''$ of the genuine eigenvector satisfy Eqs. (3.10);

b) Components $\mathbf{m}_2', \mathbf{m}_2''$ of the generalized eigenvector satisfy the following equations:

$$\begin{aligned}&\left(\gamma_1^2\mathbf{I} + \gamma_1\mathbf{N} + \mathbf{M}\right)\cdot\mathbf{m}_2' = -\left(2\gamma_1\mathbf{I} + \mathbf{N}\right)\cdot\mathbf{m}_1' \\ &\mathbf{m}_2'' = \mathbf{m}_1' + \gamma_1\mathbf{m}_2'\end{aligned} \tag{4.2}$$

c) At $c \in (0;\ c_3^{\lim})$ the matrix $\left(2\gamma_1\mathbf{I} + \mathbf{N}\right)$ is not degenerate;

d) At the same speed interval the vectors $\left(2\gamma_1\mathbf{I} + \mathbf{N}\right)\cdot\mathbf{m}_1'$ and $\mathbf{m}_1'\cdot(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})$ are orthogonal.

Proof. Conditions a) and b) flow out by direct substituting the solution (4.1) into Eq. (3.3).

To prove c) it is sufficient to demonstrate that the matrix

$$\begin{aligned}(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\cdot\left(2\gamma_1\mathbf{I} + \mathbf{N}\right) = \\ = 2\gamma_1(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\end{aligned} \tag{4.3}$$

is not degenerate. Considering multiplication of the right-hand side of (4.3) by any nonzero conjugate complex vectors $\mathbf{a}, \bar{\mathbf{a}} \in C^3$ and accounting Remark 2.1, which ensures $\operatorname{Im}(\gamma_1) \neq 0$, we arrive to

$$\begin{aligned}\operatorname{Im}\left(\mathbf{a}\cdot\left(2\gamma_1(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\right)\cdot\bar{\mathbf{a}}\right) = \\ 2\operatorname{Im}(\overline{\gamma_1})(\mathbf{a}\otimes\boldsymbol{\nu}\cdot\cdot\mathbf{C}\cdot\cdot\boldsymbol{\nu}\otimes\bar{\mathbf{a}}) \neq 0\end{aligned} \tag{4.4}$$

In obtaining (4.4) we took into consideration that $\operatorname{Im}\left(\mathbf{a}\cdot\left(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}\right)\cdot\bar{\mathbf{a}}\right) = 0$, since the matrix $\left(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}\right)$ is (real) symmetric. The last inequality in (4.4) completes the proof of condition c).

To prove d) Eq. (4.2) can be transformed into equivalent one by multiplying both sides by the nondegenerate matrix $(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})$, this gives

$$\begin{aligned}\left(\gamma_1^2(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + \gamma_1(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + (\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I})\right)\cdot\mathbf{m}_2' = \\ -\left(2\gamma_1(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\right)\cdot\mathbf{m}_1'\end{aligned} \tag{4.5}$$

Now, the vector $\mathbf{m}_1'$ belongs to the kernel space of the matrix in the left-hand side of Eq. (4.5), this flows out from Proposition 4.1.a. Moreover, the regarded matrix is complex symmetric, hence its left and right eigenvectors coincide. The latter allows us to write for the left-hand side of Eq. (4.5)

$$\mathbf{m}_1' \cdot \begin{pmatrix} \gamma_1^2(\mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu}) + \gamma_1(\mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\mathbf{\nu}) \\ +(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I}) \end{pmatrix} \cdot \mathbf{m}_2' = 0 \qquad (4.6)$$

Similarly, for the right-hand side of Eq. (4.5)

$$\mathbf{m}_1' \cdot \left(2\gamma_1(\mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu}) + (\mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\mathbf{\nu})\right)\cdot \mathbf{m}_1' = 0 \qquad (4.7)$$

In view of (3.5), Eq. (4.7) completes the proof.

COROLLARY. In the factor-space $C^3/\operatorname{Ker}\left(\gamma_1^2\mathbf{I} + \gamma_1\mathbf{N} + \mathbf{M}\right)$, the vector $\mathbf{m}_2'$ admits the following representation

$$\mathbf{m}_2' = -\left(\gamma_1^2\mathbf{I} + \gamma_1\mathbf{N} + \mathbf{M}\right)^{-1} \cdot \left(2\gamma_1\mathbf{I} + \mathbf{N}\right)\cdot \mathbf{m}_1' \qquad (4.8)$$

REMARK 4.1. At the regarded speed interval $c \in (0;\ c_3^{\lim})$ the eigenvectors of the complex symmetric matrix appearing in Eq. (4.6) may not form a set of mutually orthogonal vectors in C^3, in contrast to the mutually orthogonal eigenvectors of any real symmetric matrix.

5 DISPERSION EQUATION FOR $\mathbf{J}_6^{(\mathrm{II})}$

The traction-free boundary conditions on the surface Π_ν can be written in the form:

$$\mathbf{t}_\nu \equiv \mathbf{\nu}\cdot\mathbf{C}\cdot\cdot\nabla\mathbf{u}\Big|_{\mathbf{x}\in\Pi_\nu} = 0 \qquad (5.1)$$

Substituting the displacement field into Eq. (5.1) yields

$$\sum_{k=1}^{3} C_k \mathbf{t}_k = 0 \qquad (5.2)$$

where $\mathbf{t}_k$ are the partial surface traction.

The following two cases for the partial surface traction fields will be considered:

(i) For the Jordan normal form $\mathbf{J}_6^{(\mathrm{I})}$ and the representation (1.1), the partial surface tractions are of the form

$$\mathbf{t}_k = \left(\gamma_k\, \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu} + \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n}\right)\cdot\mathbf{m}_k'\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \qquad (5.3)$$

(ii) For the Jordan normal form $\mathbf{J}_6^{(\mathrm{II})}$ and the representation (3.8), the partial surface tractions are of the form

$$\begin{aligned} \mathbf{t}_1 &= \left(\gamma_1\, \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu} + \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n}\right)\cdot\mathbf{m}_1'\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \\ \mathbf{t}_2 &= \begin{pmatrix} \gamma_1\left(\mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu}\right)\cdot\mathbf{m}_1' + \\ \left(\gamma_1\, \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu} + \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n}\right)\cdot\mathbf{m}_2' \end{pmatrix} e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \\ \mathbf{t}_3 &= \left(\gamma_3\, \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu} + \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n}\right)\cdot\mathbf{m}_3'\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \end{aligned} \qquad (5.4)$$

Equations (5.2) can be regarded as linear system with respect to the unknown coefficients C_k. Existence of nontrivial solution of Eqs. (5.2) is equivalent to vanishing of the following determinant:

$$\mathbf{t}_1 \wedge \mathbf{t}_2 \wedge \mathbf{t}_3 = 0 \qquad (5.5)$$

Equation (5.5) provides a necessary and sufficient condition for existence of the surface wave. This equation is equally applicable to both cases (i) and (ii), since (5.5) is equivalent to linear dependence of the partial surface tractions $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$, which flows out from boundary conditions (5.2). Thus, the form of Eq. (5.5) remains unaltered when Jordan blocks arise. The only difference between cases (i) and (ii) is in representation for the partial surface tractions.

Equation (5.5) is known as the dispersion equation despite the fact, that the phase speed determined by this equation does not depend upon the wave number, or the wave frequency.

6 NON-RAYLEIGH WAVES IN CUBIC CRYSTALS

Let the unit vectors $\mathbf{e}_k$ and $k = 1, 2, 3$ be oriented along the directions of elastic symmetry for a cubic crystal, then the elasticity tensor $\mathbf{C}$ can be represented in the following form:

$$\begin{aligned} \mathbf{C} = {} & c_{11}\sum_k \mathbf{e}_k\otimes\mathbf{e}_k\otimes\mathbf{e}_k\otimes\mathbf{e}_k + \\ & c_{12}\sum_{k\neq m} \mathbf{e}_k\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_m + \\ & 4c_{44}\sum_{k<m} \operatorname{sym}(\mathbf{e}_k\otimes\mathbf{e}_m)\otimes\operatorname{sym}(\mathbf{e}_k\otimes\mathbf{e}_m), \end{aligned} \qquad (6.1)$$

where c_{11}, c_{12}, and c_{44} are the elastic constants, and

$$\operatorname{sym}(\mathbf{e}_k\otimes\mathbf{e}_m) \equiv \tfrac{1}{2}(\mathbf{e}_k\otimes\mathbf{e}_m + \mathbf{e}_m\otimes\mathbf{e}_k) \qquad (6.2)$$

The positive definite condition (2.2) applied to the elasticity tensor (6.1) gives

$$c_{11} - c_{12} > 0, \quad c_{11} + 2c_{12} > 0, \quad c_{44} > 0\,. \qquad (6.3)$$

REMARK 6.1. If $c_{11} = c_{12} + 2c_{44}$, then such an elasticity tensor corresponds to isotropic material with Lamé's constants $\lambda = c_{11}$ and $\mu = c_{44}$. For isotropic material the positive definite condition (6.3) yields the well known

$$3\lambda + 2\mu > 0, \quad \mu > 0. \tag{6.4}$$

Let the vectors $\boldsymbol{\nu}$, $\mathbf{n}$, and $\mathbf{w} = \boldsymbol{\nu} \times \mathbf{n}$ be directed along the crystallographical axes defined by the vectors $\mathbf{e}_k$ $k = 1, 2, 3$, substitution of the elasticity tensor (5.1) into Eq. (3.5) gives

$$\begin{aligned} \mathbf{M} &= \frac{c_{44} - \rho c^2}{c_{11}} \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \\ &\quad \frac{c_{11} - \rho c^2}{c_{44}} \mathbf{n} \otimes \mathbf{n} + \frac{c_{44} - \rho c^2}{c_{44}} \mathbf{w} \otimes \mathbf{w} \\ \mathbf{N} &= \frac{c_{12} + c_{44}}{c_{11}} \boldsymbol{\nu} \otimes \mathbf{n} + \frac{c_{12} + c_{44}}{c_{44}} \mathbf{n} \otimes \boldsymbol{\nu} \end{aligned} \tag{6.5}$$

The following Proposition flows out directly from the analysis of the eigenproblem for the matrix $\mathbf{R}_6$ [1]:

PROPOSITION 6.1. Suppose that the condition (6.3) is satisfied and the phase speed determined by the equation

$$\begin{aligned} \rho c^2 = &\frac{2(c_{12}+c_{44})\sqrt{c_{11}c_{44}(c_{11}+c_{12})(c_{12}+2c_{44}-c_{11})}}{(c_{11}-c_{44})^2} \\ &- \frac{(c_{11}+c_{44})(c_{11}+c_{12})(c_{12}+2c_{44}-c_{11})}{(c_{11}-c_{44})^2} \end{aligned} \tag{6.6}$$

satisfies also the equation

$$\begin{aligned} &c_{11}(c_{11} - c_{44})x^3 - 2c_{11}({c_{11}}^2 - {c_{12}}^2)x^2 + \\ &({c_{11}}^2 - {c_{12}}^2)({c_{11}}^2 - {c_{12}}^2 + 2c_{11}c_{44})x - \\ &c_{44}({c_{11}}^2 - {c_{12}}^2)^2 = 0 \end{aligned} \tag{6.7}$$

where $x = \rho c^2$, then

a) At this value of the phase speed the matrix $\mathbf{R}_6$ has the Jordan normal form ${\mathbf{J}_6}^{(\mathrm{II})}$;

b) The different roots γ_k of the Christoffel equation (2.4′) with $\operatorname{Im}\gamma_k \equiv \alpha_k < 0$ corresponding to Eq. (6.6) are as follows:

$$\begin{aligned} \gamma_1 &= -i\left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/4} \left(1 - \frac{\rho c^2}{c_{11}}\right)^{1/4}, \\ \gamma_3 &= -i\left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/2}, \end{aligned} \tag{6.8}$$

where γ_1 is the multiple root.

c) The complex amplitudes $\mathbf{m}'_k$ in the representation (3.8) have the form:

$$\begin{aligned} \mathbf{m}'_1 &= p\left(\boldsymbol{\nu}\left(1 - \frac{\rho c^2}{c_{11}}\right)^{1/4} + i\,\mathbf{n}\left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/4}\right), \\ \mathbf{m}'_2 &= sp\left(\boldsymbol{\nu} c_{44}\left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/4} + i\mathbf{n} c_{11}\left(1 - \frac{\rho c^2}{c_{11}}\right)^{1/4}\right) \\ \mathbf{m}'_3 &= \mathbf{w}, \end{aligned} \tag{6.9}$$

where p is the normalization factor:

$$p = \left(\left(1 - \frac{\rho c^2}{c_{11}}\right)^{1/2} + \left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/2}\right)^{-1/2} \tag{6.10}$$

and the parameter s is obtained by Eq. (4.8):

$$s = -c_{11} \frac{c_{11}\left(1 - \frac{\rho c^2}{c_{11}}\right)^{1/2} + c_{44}\left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/2}}{\left(c_{11}\left(1 - \frac{\rho c^2}{c_{11}}\right)^{1/2} - c_{44}\left(1 - \frac{\rho c^2}{c_{44}}\right)^{1/2}\right)^2} \tag{6.11}$$

d) The dispersion Eq. (5.5) takes the form:

$$\mathbf{t}_1 \times \mathbf{t}_2 \equiv 0, \qquad C_3 = 0 \tag{6.12}$$

REMARK 6.2. a) Direct analysis reveals that the complex amplitudes $\mathbf{m}'_k$ defined by Eq. (6.8) are not orthogonal (the case $c_{11} = c_{44}$ at which they could be orthogonal, is inconsistent with the positive definite condition and Eqs.(6.6), (6.7)).

b) Equation (6.6) defines also a transition point between decaying non-oscillating waves (genuine Rayleigh waves) and decaying oscillating waves (generalized Rayleigh waves).

Thus, Proposition 6.1 completely characterizes the surface wave propagating on a basal plane of the cubic crystal half space and corresponding to the representation (3.8).

ACKNOWLEDGMENTS

Authors thank Professor T.C.T. Ting for valuable comments and corrections, Russian Science Foundation for Fundamental Studies (Grant 04-01-00781), and Russian Academy of Sciences (Grant for Scientific Program OEMMPU No. 13) for partial financial support of one of the authors (SK).

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